Note on SLE and logarithmic CFT

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Abstract

It is discussed how stochastic evolutions may be linked to logarithmic conformal field theory. This introduces an extension of the stochastic Löwner evolutions. Based on the existence of a logarithmic null vector in an indecomposable highest-weight module of the Virasoro algebra, the representation theory of the logarithmic conformal field theory is related to entities conserved in mean under the stochastic process.

Keywords: Logarithmic conformal field theory (CFT), stochastic Löwner evolution (SLE).

1 Introduction

Stochastic Löwner evolutions (SLEs) have been introduced by Schramm [1] and further developed in [2, 3] as a mathematically rigorous way of describing certain two-dimensional systems at criticality. The method involves the study of stochastic evolutions of conformal maps, and puts into a new mathematical framework what conformal field theory (CFT) has been addressing for decades. Applications as well as formal properties and generalizations of SLE have since been investigated from various points of view. Reviews on SLE may be found in [4, 5].

The present work suggests that also logarithmic CFT (LCFT) may be linked to SLE. We refer to [6, 7, 8] for recent reviews on LCFT, and to [9] for the first systematic study of that subject. The basic observation made here is that the approach of Bauer and Bernard [10] (see also [11]) may be extended from ordinary CFT to LCFT where some primary fields are accompanied by Jordan-cell partners with new transformation properties. Other extensions of [10] may be found in [12, 13, 14, 15]. CFT and SLE-type growth processes in smaller regions of the complex plane than in ordinary chordal SLE are studied in [12]. The references [13, 14] discuss how stochastic evolutions in superspace may be linked to superconformal field theory, whereas [15] concerns the elevation to $SU(2)$ Wess-Zumino-Witten models with particular emphasis on the Sugawara construction and the Knizhnik-Zamolodchikov equations.

The initial relationship between stochastic evolutions and LCFT is mediated by stochastic differential equations and random walks on the Virasoro group. A particular scenario is outlined that extends SLE from being described by one stochastic differential equation to a system evolving according to a pair of coupled stochastic differential equations.

The relationship can be made more direct by establishing a connection between the representation theory of the LCFT and entities conserved in mean under the stochastic process. This is based on the existence of a level-two null vector in the associated indecomposable highest-weight module of the Virasoro algebra. Particular attention is paid to the module with conformal weight $\Delta = 1/4$ as it has a so-called logarithmic null vector when the central charge is $c = 1$ [16]. The description relies on the introduction of a nilpotent parameter [16, 17]. This suggests extending the regime of SLE from the complex plane to stochastic evolutions in the space obtained by augmenting the complex plane by this parameter. Alternatively, and perhaps more straightforwardly, one may interpret the result as two coupled stochastic evolutions. One of them is $\mathrm{SLE}_{\kappa=4}$ and corresponds to the SLE phase transition between simple paths (for $0 \leq \kappa \leq 4$) and self-intersecting, or rather self-touching, paths (for $4 < \kappa < 8$) defined by the so-called SLE trace. SLE_4 also corresponds to the self-dual point of a duality between SLE_κ and $\mathrm{SLE}_{16/\kappa}$ where one model is linked to the description of the boundary of the SLE hull of the other model [18, 19]. A discussion of SLE_4 itself may be found in [20].

The link between SLE and LCFT is considered in section 2, whereas section 3 contains some concluding remarks.

2 Linking SLE to LCFT

2.1 Rudiments of LCFT

A conformal Jordan cell of rank two consists of two fields: a primary field, Φ, of conformal weight Δ and its non-primary partner, Ψ, on which the Virasoro algebra generated by $\{L_n\}$ does not act diagonally. With a conventional relative normalization of the fields, we have

$$\begin{aligned} T(z)\Phi(w) &= \frac{\Delta\Phi(w)}{(z-w)^2}+\frac{\partial\Phi(w)}{z-w} \\ T(z)\Psi(w) &= \frac{\Delta\Psi(w)+\Phi(w)}{(z-w)^2}+\frac{\partial\Psi(w)}{z-w} \end{aligned} \tag{1}$$

where T is the energy-momentum tensor whose mode expansion is given in terms of $\{L_n\}$. In terms of these modes, (1) reads

$$\begin{aligned} [L_n,\Phi(z)] &= \left(z^{n+1}\partial_z+\Delta(n+1)z^n\right)\Phi(z) \\ [L_n,\Psi(z)] &= \left(z^{n+1}\partial_z+\Delta(n+1)z^n\right)\Psi(z)+(n+1)z^n\Phi(z) \end{aligned} \tag{2}$$

The two-point functions are then of the form

$$\langle\Phi(z)\Phi(w)\rangle = 0,\quad \langle\Phi(z)\Psi(w)\rangle = \frac{A}{(z-w)^{2\Delta}},\quad \langle\Psi(z)\Psi(w)\rangle = \frac{B-2A\ln(z-w)}{(z-w)^{2\Delta}} \tag{3}$$

with structure constants A and B.

The two fields transform as

$$\begin{aligned} \Phi(z) &\to (f'(z))^{\Delta}\Phi(f(z)) \\ \Psi(z) &\to (f'(z))^{\Delta}\left\{\Psi(f(z))+\ln(f'(z))\Phi(f(z))\right\} \end{aligned} \tag{4}$$

where a prime indicates a derivative with respect to the complex argument, that is, $f'(z)=\partial_z f(z)$. It was suggested by Flohr [16] to describe these transformations in a unified way by introducing a nilpotent, yet even, parameter θ satisfying $\theta^2=0$. We shall follow this idea here, though use an approach closer to the one employed in [17]. We thus define the field

$$\Upsilon(z,\theta) = \Phi(z)+\theta\Psi(z) \tag{5}$$

which is seen to be 'primary' of conformal weight $\Delta+\theta$ as it transforms like

$$\Upsilon(z,\theta) \to (f'(z))^{\Delta+\theta}\Upsilon(f(z),\theta) \tag{6}$$

This follows from the expansion

$$x^{\epsilon} = e^{\epsilon\ln(x)} = 1+\epsilon\ln(x)+\mathcal{O}(\epsilon^2) \tag{7}$$

which is exact for the nilpotent parameter $\epsilon=\theta$. The commutators (2) are now replaced by

$$[L_n,\Upsilon(z,\theta)] = \left(z^{n+1}\partial_z+(\Delta+\theta)(n+1)z^n\right)\Upsilon(z,\theta) \tag{8}$$

2.2 Stochastic differentials

We now consider the stochastic or Ito differential

$$\mathcal{G}_t^{-1}(\tau)d\mathcal{G}_t(\tau) \;=\; \alpha(\tau)dt+\beta(\tau)dB_t, \qquad \mathcal{G}_0(\tau)=1 \tag{9}$$

where we can think of $\mathcal{G}_t(\tau)$ as a τ-dependent random walk on the Virasoro group. The parameter τ is nilpotent, satisfying $\tau^2=0$, and thus of the same nature as θ introduced above. The reason for letting $\mathcal{G}_t$ depend on τ will become clear when discussing null vectors below. We are confining ourselves to one-dimensional Brownian motion, B_t, with $B_0=0$, while α and β generically are non-commutative expressions in the generators of the Virasoro algebra. It follows from $d(\mathcal{G}_t^{-1}\mathcal{G}_t)=0$ that the Ito differential of the inverse element is given by

$$d(\mathcal{G}_t^{-1})\mathcal{G}_t \;=\; (-\alpha+\beta^2)dt-\beta dB_t \tag{10}$$

We are interested in the transformations of Υ generated by $\mathcal{G}_t$. An ordinary conformal transformation (6) generated by $\mathcal{G}_t$ acts on Υ like

$$\mathcal{G}_t^{-1}\Upsilon(z,\theta)\mathcal{G}_t \;=\; \left(f_t'(z)\right)^{\Delta+\theta}\Upsilon(f_t(z),\theta) \tag{11}$$

where $f_t(z)$ is a stochastic function of z. This, however, requires that $\mathcal{G}_t$ depends trivially on τ. We shall therefore discuss the extension of (11) where the function f_t is allowed to depend on τ. It is still conformal with respect to z while the otherwise only *complex* expansion coefficients now may depend on τ as well. This modifies the transformation rule (11) as it becomes

$$\mathcal{G}_t^{-1}(\tau)\Upsilon(z,\theta)\mathcal{G}_t(\tau) \;=\; \left(f_t'(z,\tau)\right)^{\Delta+\theta}\Upsilon(f_t(z,\tau),\theta) \tag{12}$$

The function f_t may be expanded as

$$f_t(z,\tau) \;=\; h_t(z)+\tau\hat{h}_t(z) \tag{13}$$

with Ito differential

$$df_t(z,\tau) \;=\; \mu_t(z,\tau)dt+\nu_t(z,\tau)dB_t, \qquad f_0(z,\tau)=z \tag{14}$$

To relax the notation, the subscript t will occasionally be suppressed below. A goal is to compute the Ito differential of both sides of (12) and thereby relate the stochastic differential equations of $\mathcal{G}_t(\tau)$ and $f_t(z,\tau)$. It is recalled that in Ito calculus $(dB_t)^2=dt$ whereas $dtdB_t=(dt)^2=0$.

The Ito differential of the left-hand side of (12) reads

$$\begin{aligned} d\left(\mathcal{G}_t^{-1}\Upsilon\mathcal{G}_t\right) &= d(\mathcal{G}_t^{-1})\Upsilon\mathcal{G}_t+\mathcal{G}_t^{-1}\Upsilon d\mathcal{G}_t+d(\mathcal{G}_t^{-1})\Upsilon d\mathcal{G}_t \\ &= \left(-\left[\alpha_0,\mathcal{G}_t^{-1}\Upsilon\mathcal{G}_t\right]+\frac{1}{2}\left[\beta,\left[\beta,\mathcal{G}_t^{-1}\Upsilon\mathcal{G}_t\right]\right]\right)dt \\ &\quad -\left[\beta,\mathcal{G}_t^{-1}\Upsilon\mathcal{G}_t\right]dB_t \end{aligned} \tag{15}$$

where $\alpha_0 = \alpha - \frac{1}{2}\beta^2$. To facilitate the comparison we should express the differentials in the *adjoint* representation of the Virasoro algebra only. That is, α_0 and β must be linear in the generators:

$$\alpha_0(\tau) \;=\; \sum_{n\in\mathbb{Z}} a_n(\tau)L_n, \qquad\qquad \beta(\tau) \;=\; \sum_{n\in\mathbb{Z}} b_n(\tau)L_n \tag{16}$$

To avoid questions of convergence, we shall consider only finite sums, and we find

$$\begin{aligned} d\left(\mathcal{G}_t^{-1}\Upsilon(z)\mathcal{G}_t\right) &= (f')^{\Delta+\theta}\left(-[\alpha_0,\Upsilon(f)] + \frac{1}{2}[\beta,[\beta,\Upsilon(f)]]\right)dt \\ &- (f')^{\Delta+\theta}\sum_n b_n\{f^{n+1}\partial_f + (\Delta+\theta)(n+1)f^n\}\Upsilon(f)dB_t \end{aligned} \tag{17}$$

where the further evaluation is postponed till the result of a comparison of the dB_t terms can be used.

The Ito differential of the right-hand side of (12) reads

$$\begin{aligned} d\left((f')^{\Delta+\theta}\,\Upsilon(f)\right) &= (f')^{\Delta+\theta-2}\left(\frac{1}{2}(f')^2\nu^2\partial_f^2 + \{(f')^2\mu + (\Delta+\theta)f'\nu\nu'\}\partial_f \right. \\ &\qquad \left. +(\Delta+\theta)f'\mu' + \frac{1}{2}(\Delta+\theta)(\Delta+\theta-1)(\nu')^2\right)\Upsilon(f)dt \\ &+ (f')^{\Delta+\theta-1}\{f'\nu\partial_f + (\Delta+\theta)\nu'\}\Upsilon(f)dB_t \end{aligned} \tag{18}$$

It follows from a comparison of the dB_t terms that

$$f'[\beta,\Upsilon] = -\{f'\nu\partial_f + (\Delta+\theta)\nu'\}\Upsilon(f) \tag{19}$$

and hence

$$\nu_t \;=\; -\sum_n b_n f_t^{n+1} \tag{20}$$

The relation (19) also allows us to continue the evaluation of (17), and we find that

$$\mu_t \;=\; -\sum_n a_n f_t^{n+1} + \frac{1}{2}\nu_t\partial_{f_t}\nu_t \tag{21}$$

This is easily expressed in terms of f_t only, using (20), though the result is less compact than (21). Note that we get the same solution for μ_t and ν_t when considering an ordinary primary field instead of the logarithmic one in (12) (or the simpler one in (11)). This illustrates that the evaluation is *independent* of the parameter θ used in the construction of the logarithmic field $\Upsilon(z,\theta)$. This is actually required for the link to be universal and not depend explicitly on the field in question.

To summarize, we have found that the construction above establishes a general but formal link between a class of stochastic evolutions and LCFT: the Ito differentials (14) describing the evolution of the τ-dependent conformal maps are expressed in terms of the

parameters of the random walk on the Virasoro group (9) and (16), and this has been achieved via (12). The links (20) and (21) may of course be expanded with respect to τ. The result of doing this does not seem to be enlightening in its general form, and will therefore not be presented here. An example follows below, though.

The construction and link above pertain to LCFT based on rank-two Jordan cells. We believe, though, that the extension to higher rank [21] is straightforward.

The scenario resembling SLE is based on the simple situation where $a_{-2} = -2$ and $b_{-1} = \sqrt{k(\tau)}$ are the only non-vanishing coefficients:

$$\mathcal{G}_t^{-1}(\tau)d\mathcal{G}_t(\tau) \; = \; \left(-2L_{-2} + \frac{k(\tau)}{2}L_{-1}^2\right)dt + \sqrt{k(\tau)}dB_t, \qquad \mathcal{G}_0(\tau) = 1 \tag{22}$$

Here we have introduced the τ-dependent parameter

$$k(\tau) \; = \; \kappa + \tau\hat{\kappa} \tag{23}$$

The Ito differential of the associated τ-dependent conformal map reads

$$df_t(z,\tau) \; = \; \frac{2}{f_t(z,\tau)}dt - \sqrt{k(\tau)}dB_t, \qquad f_0(z,\tau) = z \tag{24}$$

Expanding this with respect to τ as in (13) leads to

$$\begin{aligned} dh_t(z) &= \frac{2}{h_t(z)}dt - \sqrt{\kappa}dB_t, \qquad h_0(z) = z \\ d\hat{h}_t(z) &= -\frac{2\hat{h}_t(z)}{h_t^2(z)}dt - \frac{\hat{\kappa}}{2\sqrt{\kappa}}dB_t, \qquad \hat{h}_0(z) = 0 \end{aligned} \tag{25}$$

In terms of the stochastic function $g_t(z) = h_t(z) + \sqrt{\kappa}B_t$, the first stochastic differential equation in (25) may be expressed as the celebrated Löwner's differential equation with Brownian motion as driving function or potential, also known as the SLE$_\kappa$ differential equation:

$$\partial_t g_t(z) \; = \; \frac{2}{g_t(z) - \sqrt{\kappa}B_t}, \qquad g_0(z) = z \tag{26}$$

This is now coupled to the differential equation

$$\partial_t \hat{g}_t(z) \; = \; \frac{-2\hat{g}_t(z) + \frac{\hat{\kappa}}{\sqrt{\kappa}}B_t}{(g_t(z) - \sqrt{\kappa}B_t)^2}, \qquad \hat{g}_0(z) = 0 \tag{27}$$

where we have introduced $\hat{g}_t(z) = \hat{h}_t(z) + \frac{\hat{\kappa}}{2\sqrt{\kappa}}B_t$. Ordinary SLE is recovered by disregarding the additional equation (27) which is possible since (26) is independent of the function $\hat{g}_t(z)$. This essentially corresponds to omitting the dependence on the nilpotent parameters altogether, which naturally reduces the considerations to ordinary CFT and SLE.

2.3 Expectation values and logarithmic null vectors

To obtain a more direct relationship between SLE and LCFT, we should link the representation theory of the conformal algebra, through the construction of logarithmic null vectors, to entities conserved in mean under the stochastic process. From the group differential (9) it follows that the time evolution of the expectation value of $\mathcal{G}_t(\tau)|\Delta+\theta\rangle$ is given by

$$\partial_t \mathbf{E}[\mathcal{G}_t(\tau)|\Delta+\theta\rangle] \;=\; \mathbf{E}[\mathcal{G}_t(\tau)\left(\alpha_0(\tau)+\frac{1}{2}\beta^2(\tau)\right)|\Delta+\theta\rangle] \tag{28}$$

Here we have introduced the notation

$$|\Delta+\theta\rangle \;=\; |\Phi\rangle+\theta|\Psi\rangle \tag{29}$$

where (cf. (1) and (2))

$$L_0|\Phi\rangle \;=\; \Delta|\Phi\rangle, \qquad\qquad L_0|\Psi\rangle \;=\; \Delta|\Psi\rangle+|\Phi\rangle \tag{30}$$

such that $L_0|\Delta+\theta\rangle = (\Delta+\theta)|\Delta+\theta\rangle$, and $L_{n>0}|\Delta+\theta\rangle = 0$. We should thus look for processes allowing us to put

$$\left(\alpha_0(\tau)+\frac{1}{2}\beta^2(\tau)\right)|\Delta+\theta\rangle \;\simeq\; 0 \tag{31}$$

in the representation theory. The expression $\mathcal{G}_t(\tau)|\Delta+\theta\rangle$ is then a so-called martingale of the stochastic process $\mathcal{G}_t(\tau)$.

Before doing that, let us indicate how time evolutions of some general expectation values may be evaluated. This extends one of the main results in [10] on ordinary SLE, and follows from the extension to the graded case discussed in [13, 14]. In this regard, observables of the process $\mathcal{G}_t$ are thought of as functions of $\mathcal{G}_t$. On such a function F, admitting a 'sufficiently convergent' Laurent expansion, we introduce the action of the vector field ∇_n as

$$(\nabla_n F)(\mathcal{G}_t) \;=\; \frac{d}{du}F(\mathcal{G}_t e^{uL_n})|_{u=0} \tag{32}$$

Referring to the notation in (16), we then have

$$\partial_t \mathbf{E}[F(\mathcal{G}_t(\tau))] \;=\; \mathbf{E}[\left(\alpha_0(\nabla,\tau)+\frac{1}{2}\beta^2(\nabla,\tau)\right)F(\mathcal{G}_t(\tau))] \tag{33}$$

with

$$\alpha_0(\nabla,\tau) \;=\; \sum_n a_n(\tau)\nabla_n, \qquad\qquad \beta(\nabla,\tau) \;=\; \sum_n b_n(\tau)\nabla_n \tag{34}$$

We now return to (31) and shall consider the situation where it corresponds to a null vector at level two in the indecomposable highest-weight module generated from $|\Delta+\theta\rangle$ [16, 17]. Such a logarithmic null vector, $|\chi(c,\Delta),\theta\rangle$, at level two (or one, see below) may be characterized by the vanishing conditions

$$L_1|\chi(c,\Delta),\theta\rangle \;=\; L_2|\chi(c,\Delta),\theta\rangle \;=\; 0 \tag{35}$$

These conditions turn out to be too restrictive for the characterization of logarithmic null vectors at levels higher than two [16, 17], but suffice for our level-two purposes. Imposing $L_{-1}|\Delta+\theta\rangle = 0$ implies $0 = L_1L_{-1}|\Delta+\theta\rangle = 2L_0|\Delta+\theta\rangle = 2(\Delta+\theta)|\Delta+\theta\rangle$, and since we naturally require Δ to be real (or at least complex), the state $L_{-1}|\Delta+\theta\rangle$ is not a logarithmic null vector. From (35) it then follows that a logarithmic null vector at level two is of the form

$$|\chi(c,\Delta),\theta\rangle \;=\; \left(-2L_{-2}+\gamma L_{-1}^2\right)|\Delta+\theta\rangle \tag{36}$$

where

$$\begin{aligned}\gamma &= \frac{3}{2\Delta+1+2\theta} = \frac{3}{2\Delta+1} - \frac{6\theta}{(2\Delta+1)^2}\\ c &= (6\gamma-8)(\Delta+\theta) = \frac{2\Delta(5-8\Delta)}{2\Delta+1} - \frac{32(\Delta-\frac{1}{4})(\Delta+\frac{5}{4})\theta}{(2\Delta+1)^2}\end{aligned} \tag{37}$$

With Δ and c independent of θ we thus conclude that a logarithmic null vector at level two exists exactly when (c,Δ) is $(1,1/4)$ or $(25,-5/4)$, in accordance with [16, 17]. A comparison of this with (31) and (22) tells us that a logarithmic null vector is constructed provided that $\tau=\theta$ and

$$k(\theta) \;=\; \frac{6}{2(\Delta+\theta)+1} \;=\; \frac{6}{2\Delta+1} - \frac{12\theta}{(2\Delta+1)^2} \tag{38}$$

We immediately recognize the announced need for a θ-dependent (or τ-dependent) walk on the Virasoro group (9). It follows from (23) and (38) that the bulk part for $\Delta = 1/4$ corresponds to $\mathrm{SLE}_{\kappa=4}$. The other conformal weight, $\Delta=-5/4$, admitting a logarithmic null vector at level two would result in a negative κ and is therefore not related immediately to ordinary SLE.

3 Conclusion

We have discussed how certain stochastic evolutions may be linked to LCFT. The method and results extend the work of Bauer and Bernard [10] and are in the vein of the graded extensions found in [13, 14]. The connection is established through the introduction of nilpotent parameters which allow one to treat the logarithmic fields in a unified way [16, 17], and to extend the realm of the conformal maps involved. Emphasis has been put on a simple scenario corresponding to a straightforward extension of SLE. The new system treats the Löwner differential equation as belonging to a pair of coupled differential equations. Ordinary SLE is recovered by disregarding the second equation. The general link is quite formal but can be made more direct by relating the representation theory of LCFT to entities conserved in mean under the stochastic process. This has been discussed explicitly in the scenario just mentioned. In a particularly interesting case, the bulk part then corresponds to SLE_4.

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Note added
After completion of the present work, the paper [22] has appeared. It also concerns the formal link between LCFT and SLE, and contains some of the results derived above, notably the case resembling SLE as based on (25) and (36).

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